ISRAEL JOURNAL OF MATHEMATICS, Vol. 21, No. 1, 1975

A TWO CARDINAL THEOREM FOR SETS OF FORMULAS IN A STABLE THEORY

BY
VICTOR HARNIK

ABSTRACT

We generalize a weak version of Lachlan's two-cardinal theorem to the context of a set of formulas of power less than λ (instead of a simple formula) and λ-compact models (instead of arbitrary models). A few applications are described.

The subject of this paper belongs to to the theory of λ-compact models. The idea is to take a theorem of (ordinary) model theory, to generalize it to a theorem concerning λ-compact models and then to infer from this new results of independent interest. This pattern first occurred in the (independent) works of Ressayre [12], Rowbottom (unpublished) and Shelah [14]. We used it jointly with Ressayre [7] and systematically pursued it in [5].

Being more specific about the method, the ordinary concept of model is replaced by that of λ-compact model and, at the same time, many other usual concepts are replaced by corresponding "λ-concepts" (e.g., "isolated typed" is replaced by "λ-isolated type", "prime model" by "λ-prime model", etc.). When $\lambda = \aleph_0$, the "λ-concepts" degenerate to the usual concepts (e.g., every model is $\aleph_0$-compact) and this is why we said above that the results in λ-compact model theory generalize ordinary results.

Our Main Lemma 3.1 is a "λ-analogue" of Lachlan's theorem in [8]. 3.1 generalizes a weak version of Lachlan's theorem (namely, the one obtained by replacing the assumption of stability by the stronger one of total trancendentality) but not Lachlan's theorem itself. The reason for this is that there seems to be no satisfactory λ-concept corresponding to the notion of prime model used by Lachlan and independently discovered by Shelah [16].

The paper is organized as follows. Sections 0–1 contain notations and a review of concepts and results from stability theory. Section 2 includes results

Received May 5, 1974

needed both for iterating the application of the Main Lemma 3.1 into the transfinite and for extending our results to all cardinals $\lambda \geqq \mu(T)$, including the singular ones. Section 3 contains the Main Lemma 3.1 which is then applied transfinitely many times to yield a two cardinal theorem for sets of less than λ formulas and λ-compact models. In Section 4 we prove a "transfer" theorem stating, roughly, that if there is a λ-compact two cardinal model then there is a λ_1-compact two cardinal model for every $\lambda, \lambda_1 \geqq \mu(T)$. In Section 5 we give a new proof of Shelah's two cardinal theorem ([13]) and in Section 6 we apply our results to minimal types and get a new proof of (an improved version of) a result of Shelah about maximally λ-saturated models (cf. [15]†. In Section 7 we briefly discuss similar results for λ-saturated rather than λ-compact models.

0. Notation

$\bar{a}$, $\bar{x}$, etc., will denote finite sequences of elements. If A is a set then $\bar{a} \in A$ will mean that all the elements of the sequence $\bar{a}$ belong to A. $\bar{a}^\frown \bar{b}$ will denote the concatenation of $\bar{a}$ and $\bar{b}$.

L will be a first order finitary language whose variables will be denoted by v, u, x with or without subscripts. Structures will be denoted by $\mathfrak{A}$, $\mathfrak{B}$, etc., and their respective universes by A, B, etc.

T will be a complete L-theory. We assume (as Shelah does e.g., in [16]) the existence of a huge model $\mathcal{M} \models T$ such that every other model of T which comes into consideration is an elementary substructure of $\mathcal{M}$. We shall write "$\models \psi$" instead of "$\mathcal{M} \models \psi$". C, D will denote subsets of M (the universe of $\mathcal{M}$). As usual (see e.g., [10]), we denote by $L(C)$ the language obtained from L by adding individual constants as names of the elements of C (we shall make no notational distinction between an element and its name) and by $F(C)$ ($F_n(C)$) the set of $L(C)$-formulas with free variable(s) $v_0(v_0, \cdots, v_{n-1})$. A type of T over C is a set $\Delta(v_0) \subset F(C)$ which is consistent in the sense that for some element $a \in M$, $\models \Delta(a)$ (i.e., $\models \phi(a)$ for all $\phi(v_0) \in \Delta(v_0)$); we say that a realizes $\Delta(v_0)$. The notion of n-type $\Delta(\bar{v}) \subset F_n(C)$ over C is defined similarly. A type over $\emptyset$ is called simply a type or a pure type. If $\Gamma(v_0)$ is a pure type and C a set then $\Gamma(C) = \{c : c \in C \text{ and } \models \Gamma(c)\}$ (an unusual notation); for a structure $\mathfrak{A}$ we write $\Gamma(\mathfrak{A}) = \Gamma(A)$.

The complete type of an element a over C is $p(a, C) = \{\phi(v_0) : \phi(v_0) \in F(C)$ and $\models \phi(a)\}$; $p(\bar{a}, C)$ will be the complete n-type of the n-tuple $\bar{a}$ over C. $S(C)$

† After completing this paper, we have learned from Prof. S. Shelah that he obtained the same improvement before us; see remark at the end of this paper.

will be the set of all complete (1–) types over C. For $q \in S(C)$, $C_1 \subset C$, let $q \mid C_1 = q \cap F(C_1)$.

We remind the reader the closely related notions of λ-saturated and λ-compact model. $\mathfrak{A}$ is λ-saturated if for every $C \subset A$ with $|C| < |A|$ and every $p \in S(C)$, p is realized by an element of A. $\mathfrak{A}$ is λ-compact if every type $\Delta(v_0)$ over A with $|\Delta(v_0)| < \lambda$ is realized by an element of A. For $\lambda > |T|$ the two notions are equivalent and for $\lambda \leqq |T|$ the notion of λ-compact is weaker than λ-saturated.

Finally, a set of indiscernibles over C is an infinite set I such that for any $n < \omega$, any two n-tuples of distinct elements of I have the same complete n-type over C.

1. Preliminaries

T is called stable in λ iff for all C, $|C| = \lambda$ implies $|S(C)| = \lambda$. T is stable if it is stable in some cardinal.

Stability can also be defined with the help of Morley trees. A *Morley Tree* of height μ is a family $\{\phi_s(v_0): s \in {}^{<\mu}2\}$ of $L(M)$-formulas such that for all $\eta \in {}^{\mu}2$, $\Phi_\eta(v_0) = \{\phi_{\eta|\alpha}(v): \alpha < \mu\}$ is a consistent type and all $s \in {}^{<\mu}2$, $\phi_{s1}(v_0) = \neg\, \phi_{s0}(v_0)$. Here, ${}^{\mu}2({}^{<\mu}2)$ is the set of all sequences of 0's and 1's of length $\mu(<\mu)$. It is easily seen that T is stable iff it does not have arbitrarily high Morley trees. For T stable, define $\mu(T)$ to be the first infinite cardinal μ such that T has no Morley tree of height μ. T is called totally transcendental if $\mu(T) = \aleph_0$ (this name is usually reserved for theories which are, in addition, countable but the author sees no reason for this restriction).

We now review some known facts about stable theories. Shelah has proven (essentially in [14], see also [15], [16]):

Proposition 1.1. *If T is stable then $\mu(T) \leqq |T|^+$.*

It is an easy exercise to show the following.

Proposition 1.2. *If T is stable and $L' \subset L$ then $T' = T/L'$, the reduct of T to L', is stable and $\mu(T') \leqq \mu(T)$.*

If $p \in S(C)$, we say that $\Delta(v_0)$ isolates p if $\Delta(v_0) \subset p$ and p is the unique extension of $\Delta(v_0)$ to a type in $S(C)$. p is λ-isolated if it is isolated by a type $\Delta(v_0)$ with $|\Delta(v_0)| < \lambda$. Ressayre [12], Rowbottom (unpublished) and Shelah [14] proved both 1.3 and 1.4 below.

PROPOSITION 1.3. *If $\Delta(v_0) \subset F(C)$ is a type then there is $\Sigma(v_0)$ such that $\Sigma(v_0) \supset \Delta(v_0)$, $|\Sigma(v_0) - \Delta(v_0)| < \mu(T)$ and $\Sigma(v_0)$ isolates a type of $S(C)$.*

The most important consequence of 1.3 is the existence of λ-prime models. A model $\mathfrak{A} \models T$ is λ-prime over C if $A \supset C$, $\mathfrak{A}$ is λ-compact and every elementary map of C into any λ-compact model $\mathfrak{B}$ can be extended to an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

THEOREM 1.4. *If T is stable and $\lambda \geqq \mu(T)$ then for every set C there is a model $\mathfrak{A} \models T$ which is λ-prime over C.*

To prove 1.4 one constructs an $\mathfrak{A}$ which is strictly λ-prime over C. By this we mean that there is enumeration such that $A = C \cup \{a_\xi\}_{\xi<\kappa}$ and for all $\xi < \kappa$, $p(a_\xi, C \cup \{a_\eta\}_{\eta<\xi})$ is λ-isolated. It is immediately seen that any strictly λ-prime model over C is λ-prime over C (the name "strictly λ-prime" has been suggested by Shelah).

$\mathfrak{A}$ is called λ-atomic over C (cf. [7]) if for any $\bar{a} \in A$, $p(\bar{a}, C)$ is λ-isolated. Let's call a cardinal λ "well-behaved" if $\lambda \geqq \mu(T)$ and for every C, if $\mathfrak{A}$ is λ-prime over C then it is λ-atomic over C. Ressayre [12] and Rowbottom proved (independently) that every regular $\lambda \geqq \mu(T)$ is well-behaved and, for $\lambda > |T|$, Shelah [16] extended this to cf $\lambda \geqq \kappa(T)$ ($\kappa(T)$ is defined in [15] and [16]; it is a cardinal which satisfies $\kappa(T) \leqq \mu(T)$; see also [6] where Shelah's proof is outlined at the end of the paper). This last condition cannot be dropped as the following example shows. Let L be a language containing unary predicates $P(v_0)$ and $P_n(v_0)$, $n < \omega$ and binary predicates $R_n(v_0, v_1)$, $n < \omega$. Let T be a theory whose axioms insure that:

(1) $\neg(P(v_0) \wedge P_n(v_0))$; $\neg(P_n(v_0) \wedge P_m(v_0))$ for all $n < m < \omega$.

(2) $R_n(v_0, v_1)$ is the graph of a function mapping (the set of elements satysfying) $P(v_0)$ into $P_n(v_0)$.

$$\text{(3)} \qquad \forall v_0 \cdots \forall v_{n-1}\Big(\bigwedge_{i<n} P_i(v_i) \rightarrow \exists u \bigwedge_{i<n} R_i(u, v_i)\Big).$$

T is easily seen to be a complete theory stable in λ iff $\lambda^{\aleph_0} = \lambda$. We have $\mu(T) = \kappa(T) = \aleph_1$. If cf $\lambda = \aleph_0$ then λ is "ill-behaved" as one can see in the following way. Let $C = \bigcup_{n<\omega} C_n$ such that $|C| = \lambda$, $|C_n| < \lambda$ and all elements of C_n satisfy $P_n(v_0)$, $n < \omega$. Then, we claim, no λ-compact model $\mathfrak{A}$ containing C can be λ-atomic. Indeed, A must contain elements $\{a_n\}_{n<\omega}$ such that $\models P_n(a_n)$ but $a_n \notin C_n$ and also must contain an element a such that $\models P(a)$ and $\models R_n(a, a_n)$ for all $n < \omega$. $p(a, C)$ is not λ-isolated!

DEFINITION 1.5. (Shelah [15]) We say that a type $p \in S(C)$ does not split over $D \subset C$ if for every $\bar{a}, \bar{b} \in C$ and every L-formula $\psi(v_0, \bar{x})$, if $p(\bar{a}, D) = p(\bar{b}, D)$ then $\psi(v_0, \bar{a}) \in p$ iff $\psi(v_0, \bar{b}) \in p$.

The importance of this notion is illustrated by the following facts.

THEOREM 1.6. (Shelah [15] and the author [5], independently). *If T is stable and $p \in S(C)$ then there is $D \subset C$, $|D| < \mu(T)$ such that p does not split over D.*

COROLLARY 1.7. *Let I be a set of indiscernibles over C. If d is an element then there exists $I_0 \subset I$, $|I_0| < \mu(T)$,* such that *$I - I_0$ is a set of indiscernibles over $C \cup \{d\}$. Consequently, if D is a set of cardinality less than λ, where $\lambda > \mu(T)$ or $\lambda = \mu(T)$ and λ regular, then there is $I_0 \subset I$, $|I_0| < \lambda$, such that $I - I_0$ is a set of indiscernibles over $C \cup D$.*

For a direct proof of 1.7 see proposition 1.3 in [7] (the same proof yields, in fact, 1.6). It should be mentioned that Shelah improved 1.7 in [15] by obtaining it with $\kappa(T)$ instead of $\mu(T)$.

PROPOSITION 1.8. *Assume T to be stable. Given sets C and $\{d_\alpha\}_{\alpha<\kappa}$, denote $p_\alpha = p(d_\alpha, C \cup \{d_\beta\}_{\beta<\alpha})$. If $p_\beta \subseteq p_\alpha$ whenever $\beta < \alpha < \kappa$ and p_α does not split over C then $\{d_\alpha\}_{\alpha<\kappa}$ is a set of indiscernibles over C.*

The principle captured by 1.8 originates with Morley [10], theorem 4.6, and has been used repeatedly by Shelah [13]–[16] and the author [5], [6].

The following beautiful definability theorem is due independently to Baldwin [1] and Shelah [16] (theorem 3.1), and was crucial in Lachlan's proof [8]. It is, in a sense, a refinement of 1.6 obtained by "localizing" that statement to ψ-types (see, e.g., [16] for a definition). This idea of "localizing" comes from Shelah's [14] and proved to be very fruitful.

THEOREM 1.9. *Assume T to be stable. If C is a set and a an element then for every L-formula $\psi(u, \bar{x})$ there is an $L(C)$-formula $\phi_\psi(\bar{x})$ such that for all $\bar{c} \in C$, $\models \psi(a, \bar{c})$* iff *$\models \phi_\psi(\bar{c})$.*

2. λ-compactness with respect to $\Gamma(v_0)$

Let T be a stable theory, λ a cardinal, $\lambda \geqq \mu(T)$ and $\Gamma(v_0)$ a pure type with $|\Gamma(v_0)| < \lambda$.

DEFINITION 2.1. A set C is called λ-compact with respect to $\Gamma(v_0)$ iff for any type $\Delta(v_0)$ over C, if $|\Delta(v_0)| < \lambda$ and $\Gamma(v_0) \subset \Delta(v_0)$ then $\Delta(v_0)$ is realized by an element of C.

The main result of this section is the following.

THEOREM 2.2. *Under the assumptions specified above, if C is λ-compact with respect to $\Gamma(v_0)$ and $\mathfrak{A}$ is a model of T which is λ-prime over C then $\Gamma(\mathfrak{A}) = \Gamma(C)$.*

PROOF. If λ is well-behaved then the theorem is trivial: Let $a \in \Gamma(\mathfrak{A})$ and let $\Delta(v_0) \subset F(C)$, $|\Delta(v_0)| < \lambda$, isolate $p(a, C)$. We may assume that $\Gamma(v_0) \subset \Delta(v_0)$ and so, there is an $a' \in C$ realizing $\Delta(v_0)$. Since $\Delta(v_0)$ isolates $p(a, C)$, it follows that $p(a', C) = p(a, C)$. This implies $a = a'$ (because $v_0 = a' \in p(a', C)$) which means that $a \in \Gamma(C)$.

We now turn to a proof of 2.2 which does not use the assumption that λ is well-behaved. At a first reading, one might skip the coming proof and go to the Main Lemma 3.1 in Section 3 and then to Sections 4–6 to see how some of the subsequent results are proved for well-behaved cardinalities λ. We first prove two lemmas the second of which will be used also in Section 3.

LEMMA 2.3. *If C is λ-compact with respect to $\Gamma(v_0)$ and a realizes a λ-isolated type over C then $C \cup \{a\}$ is λ-compact with respect to $\Gamma(v_0)$. Also, $\Gamma(C \cup \{a\}) = \Gamma(C)$.*

PROOF. Denote by $\Sigma(v_0)$ the type over c of power less than λ which isolates $p(a, C)$.

Let $\Delta(v_0) \supset F(C \cup \{a\})$ be a type extending $\Gamma(v_0)$ and such that $|\Delta(v_0)| < \lambda$. We want to show that $\Delta(v_0)$ is realized by an element of C. Obviously, $\Delta(v_0) = \Delta'(v_0, a)$ for some 2-type $\Delta'(v_0, v_1) \subset F_2(C)$. Define a new type $\Delta''(v_0) = \{\exists v_1 \psi(v_0, v_1) : \psi(v_0, v_1)$ is a finite conjunction of formulas from $\Delta'(v_0, v_1) \cup \Sigma(v_1)\}$. Then $\Gamma(v_0) \subset \Delta''(v_0) \subset F(C)$ and $|\Delta''(v_0)| < \lambda$ so, by the λ-compactness of C with respect to $\Gamma(v_0)$, there is a $c \in C$ such that $\models \Delta''(c)$. It follows easily, by the definition of $\Delta''(v_0)$, that there is an a' such that $\models \Delta'(c, a') \cup \Sigma(a')$. As $\Sigma(v_0)$ isolates $p(a, C)$, we get that $p(a', C) = p(a, C)$. Hence, $\models \Delta'(c, a')$ implies $\models \Delta'(c, a)$ i.e., $\models \Delta(c)$. So, $\Delta(v_0)$ is realized in C. We have still to show that $\Gamma(C \cup \{a\}) = \Gamma(C)$. This equality could be violated only if $a \in \Gamma(C \cup \{a\})$. But in this case, one can see that $a \in \Gamma(C)$ by the argument used to prove 2.2 for well-behaved λ.

LEMMA 2.4. *If* $\{C_\alpha\}_{\alpha<\delta}$ *is an increasing sequence of* λ*-compact sets with respect to* $\Gamma(v_0)$ *and if* $\Gamma(C_\alpha)=\Gamma(C_0)$ *for all* $\alpha<\delta$, *then* $C_\delta=\bigcup_{\alpha<\delta}C_\alpha$ *is also* λ*-compact with respect to* $\Gamma(v_0)$.

PROOF. Without loss of generality, we may assume that δ is a regular cardinal smaller than λ (this is so because for δ a successor ordinal the lemma is a triviality and for δ limit we may consider a cofinal subsequence of $\{C_\alpha\}_{\alpha<\delta}$ of length cfδ; cfδ is a regular cardinal and if cf$\delta\geqq\lambda$ then the theorem is, again, trivial).

Let $\Delta(v_0)\subset F(C_\delta)$ be a type extending $\Gamma(v_0)$ and with $|\Delta(v_0)|<\lambda$. We want to show that $\Delta(v_0)$ is realized in C_δ. At this point we may assume, without loss of generality, that $\lambda>|T|$ (if $\lambda\leqq|T|$ then we can take a sublanguage $L'\subset L,|L'|<\lambda$, such that all formulas in $\Delta(v_0)$ belong to $L'(C_\delta)$ and consider the theory $T'=T|L'$; by 1.2, T' is stable and $\mu(T')\leqq\mu(T)\leqq\lambda$; the assumption of λ-compactness with respect to $\Gamma(v_0)$ is clearly preserved when we restrict T to T'). Let $q\in S(C_0)$ be a type such that $q\cup\Delta(v_0)$ is consistent. By 1.6, there is $E\subset C_0$, $|E|<\mu(T)\leqq\lambda$ such that q does not split over E. Let $\Delta(v_0)=\bigcup_{\alpha<\delta}\Delta_\alpha(v_0)$ where $\Delta_\alpha(v_0)\subset F(C_\alpha)$. Define by induction on $\xi<\lambda$, elements $a_\xi\in C_0$ such that, if $\xi=\delta\cdot\beta+\alpha$ with $\alpha<\delta$ then a_ξ realizes $\Delta_\alpha(v_0)\cup q|(E\cup\{a_\eta\}_{\eta<\xi})$. Such an a_ξ exists in C_α because of the λ-compactness of this with respect to $\Gamma(v_0)$ (the assumption that $\lambda>|T|$ is now used to insure that the type under consideration has power $<\lambda$). Moreover, $a_\xi\in\Gamma(C_\alpha)=\Gamma(C_0)$ and so, $a_\xi\in C_0$. As

$$p_\xi=p(a_\xi,E\cup\{a_\eta\}_{\eta<\xi})=q\,|(E\cup\{a_\eta\}_{\eta<\xi})$$

and q does not split over E we see that the assumptions of 1.8 are fufilled so that we can conclude that $\{a_\xi\}_{\xi<\lambda}$ is a set of indiscernibles over E. By 1.7, there is an $I_0\subset I,|I_0|<\lambda$, such that $I-I_0$ is a set of indiscernibles over $E\cup D$, where D is the set of elements of C_δ which occur in $\Delta(v_0)$ (at this point we use again the fact that $\lambda>|T|$ to conclude, with the help of 1.1, that λ satisfies the assumptions of 1.7). Notice that for any $\alpha<\delta$, there are, by construction, λ many elements of I realizing $\Delta_\alpha(v_0)$. It follows that for every $\alpha<\delta$, some and hence all elements of $I-I_0$ realize $\Delta_\alpha(v_0)$. We conclude that $\Delta(v_0)=\bigcup_{\alpha<\delta}\Delta_\alpha(v_0)$ is realized in C_δ (by any element of $I-I_0$).

THE PROOF OF 2.2 (CONCLUDED). We may assume that $\mathfrak{A}$ is strictly λ-prime over C, so $A_\alpha=C\cup\{a_\alpha\}_{\alpha<\kappa}$ and each a_α realizes a λ-isolated type over $A_\alpha=C\cup\{a_\beta\}_{\beta<\alpha}$. By induction on α, it follows that A_α is λ-compact with respect to $\Gamma(v_0)$ and $\Gamma(A_\alpha) = \Gamma(C)$. The induction uses 2.3 for the successor case and 2.4 for the limit case. We infer that $\Gamma(\mathfrak{A}) = \Gamma(C)$.

3. The Main lemma

We now turn to the main result of the paper which, as said in the introduction, is an analogue of Lachlan's result in [8]. The proof is inspired by Lachlan's. As a matter of fact, we have in mind a simplified version of that proof due to Shelah (private communication). For a slightly different simplified proof see Baldwin [2].

Main Lemma 3.1. *Let T be a stable theory, $\lambda \geqq \mu(T)$ and $\bar{\Gamma}(v_0)^-$ a type of T with $|\Gamma(v_0)| < \lambda$. If T has a pair of λ-compact models $\mathfrak{A}_0$, $\mathfrak{A}_1$ such that $\mathfrak{A}_0 \underset{\neq}{<} \mathfrak{A}_1$ and $\Gamma(\mathfrak{A}_1) = \Gamma(\mathfrak{A}_0)$ then there exists a λ-compact model $\mathfrak{A}_2$ such that $\mathfrak{A}_1 \underset{\neq}{<} \mathfrak{A}_2$* and $\Gamma(\mathfrak{A}_2) = \Gamma(\mathfrak{A}_0)$.

Proof. Pick $c_0 \in A_1 - A_0$. By 1.9, for every L-formula $\psi(u, \bar{x})$ there is an $L(A_0)$-formula $\phi_\psi(\bar{x})$ such that for $\bar{a} \in A_0$, $\models \psi(c_0, \bar{a})$ iff $\models \phi_\psi(\bar{a})$. Define $q = \{\psi(v_0, \bar{b})\colon \psi(u, \bar{x})$ an L-formula, $\bar{b} \in A_1$, $\models \phi_\psi(\bar{b})\}$. It is easy to see that $q \in S(A_1)$, q does not split over A_0 and $p(c_0, A_0) \subset q$. Let c_1 be an element realizing q. Obviously, c_1 does not realize $\Gamma(v_0)$, so $\Gamma(A_1 \cup \{c_1\}) = \Gamma(\mathfrak{A}_1) = \Gamma(\mathfrak{A}_0)$.

Claim 3.2. *$A_1 \cup \{c_1\}$ is λ-compact with respect to $\Gamma(v_0)$.*

Proof of 3.2. Let $\Delta(v_0)$ be a type over $A_1 \cup \{c_1\}$ extending $\Gamma(v_0)$ and with $|\Delta(v_0)| < \lambda$. The formulas of $\Delta(v_0)$ are of the form $\psi(c_1, v_0, \bar{b})$ with $\psi(u, v_0, \bar{x})$ an L-formula and $\bar{b} \in A_1$. We want to show the existence of an element $a \in A_1$ realizing $\Delta(v_0)$. By the very definition of c_1, this is equivalent to finding an $a \in A_1$ which satisfies the set $\Delta'(v_0) \subset F(A_1)$ of formulas defined by $\Delta'(v_0) = \{\phi_\psi(v_0, \bar{b})\colon \psi(c_1, v_0, \bar{b}) \in \Delta(v_0)\}$. Because of the λ-compactness of $\mathfrak{A}_1$, all we have to do is to prove that $\Delta'(v_0)$ is a type. Assume that this is not the case. Then for some $k < \omega$ and some formulas $\psi_i(c_1, v_0, \bar{b}_i) \in \Delta(v_0)$, $1 \leqq i \leqq k$,

$$(1) \qquad \models \neg \exists v_0 \bigwedge_{i=1}^{k} \phi_{\psi_i}(v_0, \bar{b}).$$

On the other hand, the $\bar{b}_i$'s satisfy, for every finite conjunction $\gamma(v_0)$ of formulas from $\Gamma(v_0)$,

$$\models \exists v_0\Big(\bigwedge_{i=1}^{k} \psi_i(c_1, v_0, \bar{b}_i) \wedge \gamma(v_0)\Big).$$

Denote this last sentence satisfied by c_1 and the $\bar{b}_i$'s by $\gamma^*(c_1, \cdots, \bar{b}_k)$. We obviously have that

$$\text{(2)} \qquad \models \phi_{\gamma^*}(\bar{b}_1, \cdots, \bar{b}_k) \qquad \text{for all } \gamma(v_0) \text{ as above.}$$

Conditions (1) and (2) specify $< \lambda\ L(A_0)$-formulas satisfied by $\bar{b}_1, \cdots, \bar{b}_k$. Because of the λ-compactness of $\mathfrak{A}_0$, there are $\bar{a}_1, \cdots, \bar{a}_k \in A_0$ satisfying the same formulas, i.e.,

$$\text{(1')} \qquad \models \neg \exists_0 \bigwedge_{i=1}^{k} \phi_{\psi_i}(v_0, \bar{a}) \quad \text{and}$$

$$\text{(2')} \qquad \models \phi_{\gamma^*}(\bar{a}_1, \cdots, \bar{a}_k)$$

for every $\gamma(v_0)$ as above. By the definition of ϕ_{γ^*}, (2') implies that $\models \gamma^*(c_0, \bar{a}_1, \cdots, \bar{a}_k)$ i.e.,

$$\models \exists v_0 \Big(\bigwedge_{i=1}^{k} \psi_i(c_0, v_0, \bar{a}_i) \wedge \gamma(v_0)\Big).$$

As this is true for every finite conjunction $\gamma(v_0)$ of $\Gamma(v_0)$-formulas, it follows that

$$\Gamma(v_0) \cup \Big\{ \bigwedge_{i=1}^{k} \psi_i(c_0, v_0, \bar{a}_i)\Big\}$$

is a consistent type and hence, by the λ-compactness of $\mathfrak{A}_1$, it is realized by an element $a \in A_1$. As $a \in \Gamma(\mathfrak{A}_1) = \Gamma(\mathfrak{A}_0)$ we have, in fact, that $a \in A_0$. As $\models \psi_i(c_0, a, \bar{a}_i)$, we have that $\models \phi_{\psi_i}(a, \bar{a}_i)$ for $1 \leqq i \leqq k$. This contradicts (1') concluding the proof of 3.2.

To finish the proof of 3.1, take $\mathfrak{A}_2$ to be a model of T which is λ-prime over $A_1 \cup \{c_1\}$. By 2.2,

$$\Gamma(\mathfrak{A}_2) = \Gamma(A_1 \cup \{c_1\}) = \Gamma(\mathfrak{A}_0).$$

REMARK. We would like to point out the relationship between 3.1 and Lachlan's result which inspired it. For $\lambda = \mu(T) = \aleph_0$, 3.1 yields the following: *If* $\mathfrak{A}_0$, *and* $\mathfrak{A}_1$ *are models of a (possibly uncountable) totally transcendental theory T such that* $\mathfrak{A}_0 \precneqq \mathfrak{A}_1$ *and* $Q(\mathfrak{A}_1) = Q(\mathfrak{A}_0)$, $Q(\mathfrak{A}_0)$ *a unary predicate, then there is a model* $\mathfrak{A}_2$ *of T such that* $\mathfrak{A}_1 \precneqq \mathfrak{A}_2$ *and* $Q(\mathfrak{A}_2) = Q(\mathfrak{A}_0)$. Lanchlan's theorem is the similar statement for countable stable theories. Thus, 3.1 only partially generalizes Lachlan's result. The question whether Lachlan's theorem is true for arbitrary stable theories is still open (but see Section 5 below).

We can now iterate applications of 3.1 as many times as we wish using 2.4 at limit stages.

COROLLARY 3.3. *If* $\mathfrak{A}_0$, $\mathfrak{A}_1$, *are* λ*-compact models of a stable theory T such that* $\mathfrak{A}_0 \precneqq \mathfrak{A}_1$ *and* $\Gamma(\mathfrak{A}_1) = \Gamma(\mathfrak{A}_0)$ *where* $\lambda \geqq \mu(T)$ *and* $\lambda > |\Gamma(v_0)|$, *then we can find* λ*-compact models* $\mathfrak{A}_\alpha$ $\alpha \geqq 2$, *such that* $\mathfrak{A}_\beta \precneqq \mathfrak{A}_\alpha$ *whenever* $0 \leqq \beta < \alpha$ *and* $\Gamma(\mathfrak{A}_\alpha) = \Gamma(\mathfrak{A}_0)$ *for all* α. *Thus,* $\mathfrak{A}_1$ *has arbitrarily large elementary extensions* $\mathfrak{A}$ *with* $\Gamma(\mathfrak{A}) = \Gamma(\mathfrak{A}_0)$.

PROOF. Define $\mathfrak{A}_\alpha$ by induction on α. If $\mathfrak{A}_\alpha$ is defined, get $\mathfrak{A}_{\alpha+1}$ by applying 3.1 to the couple of models $\mathfrak{A}_0$, $\mathfrak{A}_\alpha$; if $\mathfrak{A}_\alpha$ is defined for all $\alpha < \delta$, δ limit, then 2.4 implies that $\bigcup_{\alpha<\delta} A_\alpha$ is λ-compact with respect to $\Gamma(v_0)$ and so, we can take $\mathfrak{A}_\delta$ to be λ-prime over that set.

4. From λ-compact to λ_1-compact models

This section (as well as the subsequent ones) does not depend on Corollary 3.3 and thus, for well-behaved λ and λ_1, does not depend on 2.4.

Given λ-compact models of T, $\mathfrak{A}_0$ and $\mathfrak{A}_1$, such that $\mathfrak{A}_0 \precneqq \mathfrak{A}_1$ and $\Gamma(\mathfrak{A}_0) = \Gamma(\mathfrak{A}_1)$, we may iterate the application of the Main Lemma 3.1 ω times without encountering any difficulty. We get a strictly increasing elementary chain $\{\mathfrak{A}_n\}_{n<\omega}$ of λ-compact models of T such that $\Gamma(\mathfrak{A}_n) = \Gamma(\mathfrak{A}_0)$ for all $n < \omega$. Once $\mathfrak{A}_n$ is defined, the construction of $\mathfrak{A}_{n+1}$ by the process described in the proof of 3.1 can be carried out in many ways since it depends on the arbitrary choice of $c_0 \in A_n - A_0$. However, there is one obvious way of constructing the chain $\{\mathfrak{A}_n\}_{n<\omega}$ which involves just one arbitrary choice. This way yields the following result.

LEMMA 4.1. *If* $\mathfrak{A}_0 \precneqq \mathfrak{A}_1$ *are* λ*-compact models of a stable theory T,* $\lambda \geqq \mu(T)$ *such that* $\Gamma(\mathfrak{A}_1) = \Gamma(\mathfrak{A}_0)$ *where* $|\Gamma(v_0)| < \lambda$, *then there exists a model* $\mathfrak{A}$ *of T such that;*

(i) $\mathfrak{A}_0 < \mathfrak{A}$ *and* $\Gamma(\mathfrak{A}) = \Gamma(\mathfrak{A}_0)$;

(ii) *there is a sequence* $\{c_n\}_{n<\omega}$, $c_n \in A$, *of indiscernibles over* A_0 *such that for any* $n < \omega$, $A_0 \cup \{c_i\}_{i<n}$ *is included in a* λ*-compact elementary substructure of* $\mathfrak{A}$.

PROOF. Pick $c_0 \in A_1 - A_0$ and an $L(A_0)$-formula $\phi_\psi(\bar{x})$ against each L-formula $\psi(u,\bar{x})$ as in the proof of 3.1. Define by induction $\mathfrak{A}_n$ and $c_{n-1} \in A_n$ for $n \geqq 2$ such that $\mathfrak{A}_{n+1}$ is λ-prime over $A_n \cup \{c_n\}$ and c_n satisfies the type $q_n \in S(A_n)$ defined by $q_n = \{\psi(v_0,\bar{b})\colon \psi(u,\bar{x})$ an L-formula, $\bar{b} \in A_n, \models \phi_\psi(\bar{b})\}$. By the proof of 3.1 we have that $\{\mathfrak{A}_n\}_{n<\omega}$ is a strictly increasing elementary chain of λ-compact models $\Gamma(\mathfrak{A}_n) = \Gamma(\mathfrak{A}_0)$ for all $n < \omega$. Taking $\mathfrak{A} = \bigcup_{n<\omega} \mathfrak{A}_n$ we see at once that (i) is satisfied. It is easily seen that $q_n \subseteq q_m$ for $1 \leqq n \leqq m$ and that q_n does not split over A_0. A fortiori, the types

$$p_n = p(c_n, A_0 \cup \{c_i\}_{i<n}),$$

$n < \omega$, satisfy the same properties hence, by 1.8, $\{c_n\}_{n<\omega}$ is a set of indiscernibles over A_0. Notice that $\{c_i\}_{i<n}$ is contained in the λ-compact model $\mathfrak{A}_n$. This completes the proof of 4.1.

The main result of this section is the following "transfer" theorem:

THEOREM 4.2. *If* $\mathfrak{A}_0, \mathfrak{A}_1$ *are* λ*-compact models of a stable theory* T *such that* $\mathfrak{A}_0 \precneqq \mathfrak{A}_1$, $\Gamma(\mathfrak{A}_1) = \Gamma(\mathfrak{A}_0)$ *and if* $\lambda, \lambda_1 \geqq \mu(T)$, $\lambda, \lambda_1 > |\Gamma(v_0)|$, *then for every cardinal* $\kappa > \lambda_1$ *there are* λ_1*-compact models* $\mathfrak{B}_0, \mathfrak{B}_1 \models T$ *such that* $\mathfrak{B}_0 \precneqq \mathfrak{B}_1$, $\Gamma(\mathfrak{B}_1) = \Gamma(\mathfrak{B}_0)$ *and* $|\mathfrak{B}_0| \leqq \lambda_1^{|T|}$, $|\mathfrak{B}_1| \geqq \kappa$.

PROOF. Let $\mathfrak{A}$ and $\{c_n\}_{n<\omega}$ be like in the conclusion of 4.1. A standard use of the compactness theorem yields a λ_1-compact model $\mathfrak{B}_0$ of T containing a set $\{d_\alpha\}_{\alpha<\lambda_1}$ of indiscernibles such that any k-tuple of distinct elements from $\{d_\alpha\}_{\alpha<\lambda_1}$ satisfies the same L-formulas as any k-tuple of distinct elements from $\{c_n\}_{n<\omega}$. Without loss of generality, we may assume that $\mathfrak{B}_0$ is λ_1-prime over $\Gamma(\mathfrak{B}_0) \cup \{d_\alpha\}_{\alpha<\lambda_1}$ and hence $|\mathfrak{B}_0| \leqq \lambda_1^{|T|}$ (because, by Shelah [14], T is stable in $\lambda_1^{|T|}$ and by [6] it has a saturated model in that power).

We *claim* that $\{d_\alpha\}_{\alpha<\lambda_1}$ is a set of indiscernibles over $\Gamma(\mathfrak{B}_0)$. If this not true then there are k-tuples $\bar{d}, \bar{d}'$ from $\{d_\alpha\}_{\alpha<\lambda_1}$, an L-formula $\psi(\bar{x},\bar{v})$ with $k+l$ free variables and an l-tuple $\bar{b} \in \Gamma(\mathfrak{B}_0)$ such that $\models \psi(\bar{d},\bar{b}) \not\leftrightarrow \psi(\bar{d}',\bar{b})$. In other words, the l-tuple $\bar{b}$ satisfies the following type, where $\bar{v} = \langle v_0, \cdots v_{l-1}\rangle$,

$$\Sigma(\bar{v}) = \{\psi(\bar{d},\bar{v}) \not\leftrightarrow \psi(\bar{d}',v)\} \cup \Gamma(v_0) \cup \cdots \cup \Gamma(v_{l-1}).$$

Let $\bar{c}$ and $\bar{c}'$ be k-tuples from $\{c_n\}_{n<\omega}$ such that the $2k$-tuples $\bar{c}^\frown\bar{c}'$ and $\bar{d}^\frown\bar{d}'$ satisfy the same L-formulas. It follows that the set of $L(\bar{c}^\frown\bar{c}')$-formulas

$$\Sigma_1(\bar{v}) = \{\psi(\bar{c},\bar{v}) \not\leftrightarrow \psi(\bar{c}',\bar{v})\} \cup \Gamma(v_0) \cup \cdots \cup \Gamma(v_{l-1})$$

is a consistent type. As $|\Sigma_1(\bar{v})| < \lambda$ and $\bar{c}^\frown\bar{c}'$ belongs to a λ-compact substructure of $\mathfrak{A}$, the type $\Sigma_1(\bar{v})$ is realized by an l-tuple $\bar{a} \in A$. In fact, $\bar{a} \in \Gamma(\mathfrak{A}) = \Gamma(\mathfrak{A}_0)$. This shows that $\{c_n\}_{n<\omega}$ is not a set of indiscernibles over A_0, a contradiction to (ii) of 4.1.

To end the proof, consider an extension $\{d_\alpha\}_{\alpha<\kappa}$ of $\{d_\alpha\}_{\alpha<\lambda_1}$ to a set of indiscernibles over $\Gamma(\mathfrak{B}_0)$ of cardinality κ. We now show that $C = \Gamma(\mathfrak{B}_0) \cup \{d_\alpha\}_{\alpha<\kappa}$ is λ_1-compact with respect to $\Gamma(v_0)$. Let $\Delta(v_0) \subset F(C)$ be a type extending $\Gamma(v_0)$ and with $|\Delta(v_0)| < \lambda_1$. Let D be the set of d_α's occurring in $\Delta(v_0)$ and let f: $D \to \{d_\alpha\}_{\alpha<\lambda_1}$ be a bijection of D onto $D' = f(D)$. The set of formulas $\Delta'(v_0)$, obtained from $\Delta(v_0)$ by substituting each occurrence of any $d_\alpha \in D$ by an occurrence of $f(d_\alpha)$, is a consistent type. As $|\Delta'(v_0)| < \lambda_1$, $\Delta'(v_0) \subset F(B_0)$ and $\mathfrak{B}_0$ is λ_1-compact, the type $\Delta'(v_0)$ is realized by an element $b \in B_0$. In fact, $b \in \Gamma(\mathfrak{B}_0) \subset C$. Using the fact that $\{d_\alpha\}_{\alpha<\kappa}$ is a set of indiscernibles over $\Gamma(\mathfrak{B}_0)$ we conclude easily that b satisfies $\Delta(v_0)$ as well. Thus, C is indeed λ_1-compact with respect to $\Gamma(v_0)$. Take $\mathfrak{B}_1$ to be a model which is λ_1-prime over C. Then $|\mathfrak{B}_1| \geqq \kappa$, $\mathfrak{B}_1$ is λ_1-compact and $\Gamma(\mathfrak{B}_1) = \Gamma(C) = \Gamma(\mathfrak{B}_0)$. As $\mathfrak{B}_0$ is λ_1-prime over $\Gamma(\mathfrak{B}_0) \cup \{d_\alpha\}_{\alpha<\lambda_1} \subset B_1$, we may assume that $\mathfrak{B}_0 < \mathfrak{B}_1$.

The proof is now complete.

Notice that all the elements d_α, $\alpha < \kappa$, realize the same type as c_0 does. This remark, coupled with the fact that c_0 was a completely arbitrary element of $A_1 - A_0$, yields the following stronger version of 4.2.

THEOREM 4.2'. *If* $\mathfrak{A}_0$, $\mathfrak{A}_1$ *satisfy the assumption of* 4.2 *and, in addition,* $\Sigma(\mathfrak{A}_1) \neq \Sigma\mathfrak{A}_0)$ *for some type* $\Sigma(v_0)$ *then* T *has* λ_1*-compact models* $\mathfrak{B}_0$, $\mathfrak{B}_1$ *such that*

$$\mathfrak{B}_0 < \mathfrak{B}_1,\ \Gamma(\mathfrak{B}_1) = \Gamma(\mathfrak{B}_0),\ |\mathfrak{B}_0| \leqq \lambda_1^{|T|}$$

and $|\Sigma(\mathfrak{A}_1)| \geqq \kappa$.

5. Shelah's two cardinal theorem

As a corollary to 4.2, we get the following theorem of Shelah (which predates Lachlan's [8]; for countable T, Lachlan's theorem is stronger).

THEOREM 5.1. (Shelah [13]) *If* T *is stable*, $\mathfrak{A}_0, \mathfrak{A}_1 \models T$, $\mathfrak{A}_0 \underset{\neq}{<} \mathfrak{A}_1$ *and, for a unary predicate* $Q(v_0)$, $Q(\mathfrak{A}_1) = Q(\mathfrak{A}_0)$, $|Q(\mathfrak{A}_0)| \geqq \aleph_0$ *then for any* κ, μ *such that* $\kappa > \mu \geqq |T|$, T *has a model* $\mathfrak{B}$ *with* $|\mathfrak{B}| = \kappa$ *and* $|Q(\mathfrak{B})| = \mu$.

PROOF. It is a simple exercise to show that under the given assumptions, T has a pair of λ-compact models satisfying the same conditions as $\mathfrak{A}_0$, $\mathfrak{A}_1$ do. The conclusion follows then from 4.2 whenever $\mu^{|T|} = \mu$. The full conclusion follows from Vaught's theorem for two cardinals far apart [17].

REMARK 1. Shelah's original proof of this theorem in [13] was set theoretical (it involved the use of an instance of *GCH* which was then eliminated by an absoluteness argument). Subsequently, he found a purely model theoretic proof (outlined in [16]). Our proof is different from both of Shelah's. Another proof which is closer to Shelah's original one but model theoretic in nature can be given using 2.4 in the following way: We first take a couple of saturated models $\mathfrak{B}_0$, $\mathfrak{B}_1$, of a power $\lambda > |T|$ in which T is stable, such that $\mathfrak{B}_0 \underset{\neq}{<} \mathfrak{B}_1$ and $Q(\mathfrak{B}_1) = Q(\mathfrak{B}_0)$ (this is possible for any λ as mentioned and particularly easy when λ is regular). Next we construct a strictly increasing elementary tower $\{\mathfrak{B}_\alpha\}_{\alpha<\lambda^+}$ of saturated models as Chang's proof [4] using 2.4 at limit stages. This yields a model of type (λ^+, λ) where T is stable in λ. The conclusion now follows by standard techniques of stability theory. Still another proof of 5.1 can be given using Lachlan's theorem, compactness and the fact that the reduct of a stable theory to any sublanguage is stable.

REMARK 2. In analogy with 4.2′, one can show that if $\mathfrak{A}_0$, $\mathfrak{A}_1$ from 5.1 satisfy the additional hypothesis that $\Sigma(\mathfrak{A}_1) \neq \Sigma(\mathfrak{A}_0)$ for some type of $\Sigma(v_0)$ then there is a $\mathfrak{B}$ with $|Q(\mathfrak{B})| = \mu$ and $|\Sigma(\mathfrak{B})| = \kappa$. The same conclusion could be drawn by (slight modification of) either Shelah's original proof in [13] or any of the last two proofs outlined in the previous remark.

This stronger form of the two cardinal theorem is by no means a specialty of stable theories. For example (again, a slight modification of) Vaught's proof yields the following form of Vaught's two cardinal theorem in [11]: if T has a pair of models $\mathfrak{A}_0$, $\mathfrak{A}_1$ such that $\mathfrak{A}_0 < \mathfrak{A}_1$, $Q(\mathfrak{A}_1) = Q(\mathfrak{A}_0)$, $|Q(\mathfrak{A}_0)| \geqq \aleph_0$ and $\Sigma(\mathfrak{A}_1) \neq \Sigma(\mathfrak{A}_0)$ then T has a model $\mathfrak{B}$ with $|Q(\mathfrak{B})| = \aleph_0$ and $|\Sigma(\mathfrak{B}| = \aleph_1$. A similar remark applies to Chang's two cardinal theorem [4].

6. Application to minimal types

Let us remind the reader that a type $\Gamma(v_0)$ over a set C is called algebraic iff $\Gamma(v_0)$ is realized by only finitely many elements (of $\mathcal{M}$). Equivalently, $\Gamma(v_0)$ is algebraic iff there is a finite conjunction $\gamma(v_0)$ of formulas from $\Gamma(v_0)$ such that for some $n < \omega$, $\models \exists^{\leq n} v_0 \gamma(v_0)$ (i.e., "there are at most n elements v_0 such that $\gamma(v_0)$").

A type $I(v_0)$ over any set is called *minimal* if it is not algebraic and for every formula $\psi(v_0, \bar{a})$ (with $\bar{a}$ not necessarily occurring in $I(v_0)$), either $I(v_0) \cup \{\psi(v_0, \bar{a})\}$ or $I(v_0) \cup \{\neg \psi(v_0, \bar{a})\}$ is algebraic.

The notion of minimal type has been introduced by Ressayre ([7]) as a generalization of Marsh's notion of strongly minimal formula ([9], cf. also [3]). The relevance of minimal types to our subject lies in lemma 6.1 below.

A set C is algebraically closed if every algebraic type over C is realized by an element of C. If $\mathfrak{A} \models T$ then A is algebraically closed.

LEMMA 6.1. *If C is algebraically closed, $I(v_0) \subset F(C)$ a minimal type, $|I(v_0)| < \lambda$, and $|I(C)| \geqq \lambda$ then C is λ-compact with respect to $I(v_0)$.*

PROOF. Let $\Delta(v_0) \subset F(C)$, $|\Delta(v_0)| < \lambda$, be a type extending $I(v_0)$. By our assumption on C, if $\Delta(v_0)$ is algebraic then it is realized in C. If $\Delta(v_0)$ is not algebraic then, by the definition of minimality, $I(v_0) \cup \{\neg \psi(v_0)\}$ is algebraic for all $\psi(v_0) \in \Delta(v_0)$. This implies that each such $\psi(v_0)$ is satisfied by all but finitely many elements of $I(\mathfrak{A})$. Hence, $\Delta(v_0)$ is realized by all but less than λ elements of $I(\mathfrak{A})$.

It follows that if $\mathfrak{A}$ is a model with $|I(\mathfrak{A})| = \lambda \geqq \mu(T)$, $I(v_0)$ a minimal type with $|I(v_0)| < \lambda$, and if $\mathfrak{B}$ is a model λ-prime over A then $I(\mathfrak{B}) = I(\mathfrak{A})$ and, hence, $|I(\mathfrak{B})| = \lambda$. This remark allows us to improve 4.2 for the particular case of minimal types.

THEOREM 6.2. *If $\mathfrak{A}_0, \mathfrak{A}_1$ are λ-compact models of a stable theory T, $\mathfrak{A}_0 \precneqq \mathfrak{A}_1$ and $I(\mathfrak{A}_1) = I(\mathfrak{A}_0)$ where $I(v_0)$ is a minimal type with $|I(v_0)| < \lambda, \lambda_1$ and if $\lambda, \lambda_1 \geqq \mu(T)$, $\lambda_1 \geqq |T|$ then for every cardinal $\kappa > \lambda$, there is a λ_1-compact model $\mathfrak{B}_1 \models T$ such that $|\mathfrak{B}_1| = \kappa$ and $|I(\mathfrak{B}_1)| = \lambda_1$ (thus, $\mathfrak{B}_1$ is not λ_1^+-compact).*

PROOF. As in the proof of 4.2, we first construct a model $\mathfrak{B}_0$ which is λ_1-compact and contains a set $\{d_\alpha\}_{\alpha<\lambda_1}$ of indiscernibles over $I(\mathfrak{B}_0)$ but this time we take $\mathfrak{B}_0$ to be a λ_1-prime over B_0' where $\mathfrak{B}_0'$ is a model of cardinality λ_1 containing $\{d_\alpha\}_{\alpha<\lambda_1}$. By the previous remark, $|I(\mathfrak{B}_0)| = \lambda_1$ (although $|\mathfrak{B}_0|$ may be as high as $\lambda_1^{|T|}$). We next get $\mathfrak{B}_1$ as in the proof of 4.2.

If T is stable then minimal types are at hand by the following statement which we recall from [7]:

PROPOSITION 6.3. *If T is stable then there exists a type $I(v_0)$ over a set C such that $I(v_0)$ is minimal and $|I(v_0)| \leqq |T|$ (in fact, $I(v_0) < \mu(T)$).*

We illustrate now a use of this and the previous result. S. Shelah proved the following important theorem in [15]:

THEOREM 6.4. *If a theory T has a model $\mathfrak{A}$ with $|\mathfrak{A}| > \lambda^{|T|}$ which is $|T|^+$-saturated but not λ^+-saturated then for every regular cardinal λ_1, T has models of arbitrarily large cardinalities which are λ_1-saturated but not λ_1^+-saturated.*

Shelah's proof runs as follows. First he shows that if T is unstable then the conclusion of the theorem always holds. This is done by an extremely beautiful argument (theorem 6.1 in [15]) using the fact that T has arbitrarily high Morley trees. Then 6.4 is proved under the assumption that T is stable, the proof breaking down into several cases for λ_1. We now point out that the case $\lambda_1 > |T|$ easily follows for arbitrary (i.e., not necessarily regular) λ_1 from 6.1–6.3. More precisely, we prove the following.

THEOREM 6.5. *If T is stable and has a model $\mathfrak{A}$ with $|\mathfrak{A}| > \lambda^{|T|}$ which is $|T|^+$-saturated but not λ^+-saturated then for every $\lambda_1 > |T|$, T has models of arbitrarily large cardinalities which are λ_1-saturated but not λ_1^+-saturated.*

PROOF. By 6.3 there is C, $|C| \leqq |T|$ and a type $I(v_0)$ over C such that $I(v_0)$ is minimal. As $\mathfrak{A}$ is $|T|^+$-saturated, we may assume that $C \subset A$. By considering $T(C) = \mathrm{Th}(\mathfrak{A}, c)_{c \in C}$ instead of T we may assume, without loss of generality, that $C = \varnothing$.

If $|I(\mathfrak{A})| \leqq \lambda$ then one can take $\mathfrak{A}_0$, $\mathfrak{A}_0 < \mathfrak{A}$, such that $I(\mathfrak{A}) \subset A_0$ (hence, $I(\mathfrak{A}) = I(\mathfrak{A}_0)$), $\mathfrak{A}_0$ is $|T|^+$-saturated and $|\mathfrak{A}_0| \leqq (|T|^+)^{|T|} \leqq \lambda^{|T|}$. It follows that $\mathfrak{A}_0 \neq \mathfrak{A}$. The premises of 6.2 are thus satisfied by $\mathfrak{A}_0$, $\mathfrak{A}$ and the desired conclusion follows from that theorem.

If $|I(\mathfrak{A})| > \lambda$ then, by 6.1, A is λ-compact with respect to $I(v_0)$. Take $\mathfrak{A}_1$, $\mathfrak{A} < \mathfrak{A}_1$, to be a model which is λ^+-prime over A. Then $I(\mathfrak{A}_1) = I(\mathfrak{A})$, by 2.2, and $\mathfrak{A} \neq \mathfrak{A}$ as $\mathfrak{A}_1$ is λ^+-satured and $\mathfrak{A}$ is not. The premises of 6.2 are again satisfied, this time by $\mathfrak{A}$, $\mathfrak{A}_1$, and the theorem follows.

REMARK 1. By a more refined proof we could draw the same conclusion by assuming only that $\mathfrak{A}$ is $(\mu(T) + \aleph_1)$-compact (rather than $|T|^+$-saturated). Shelah has shown that $\mathfrak{A}$ $\kappa(T)^+$-saturated or, if $\mathrm{cf}\, \kappa(T) > \omega$, even $\kappa(T)$-saturated is also enogh.

REMARK 2. In the present context, we had to restrict ourselves to $\lambda_1 > |T|$ as we are dealing with the notion of λ_1-compactness which is weaker than

λ_1-saturability if $\lambda_1 \leqq |T|$. In the next section we discuss the problem of proving the results of this paper for "λ-saturated" instead of "λ-compact" and indicate how, in certain cases, our proof of 6.5 can be extended to values of λ_1 smaller than $|T|^+$.

7. λ-saturated vs. λ-compact

As the notions of a λ-saturated and a λ-compact structure are not equivalent, it makes sense to try and define notions such as λ-prime and λ-isolated which are suitable for the concept of λ-saturated rather than λ-compact. This has been done (see [16] where these were called $(\lambda, 1)$-prime and $(\lambda, 1)$-isolated). Of course, these notions are different, for $\lambda \leqq |T|$, from the ones we used in Sections 1–6. Most of the results of this paper can be proven for the new notions as well. We now briefly indicate some differences.

Lemma 2.4 and, hence, all the subsequent results, seem to work only for $\lambda \geqq \mu'(T)$ where $\mu'(T) = \mu(T)^+$ if $\mu(T) = \kappa(T)$ and both are singular and $\mu'(T) = \mu(T)$ otherwise ($\kappa(T)$ comes in through Shelah's improvement of 1.7 metioned in Section 1). The Main Lemma 3.1 can be proven, for λ-saturated models, using 1.6 instead of the more delicate 1.9 (in fact, the delicacy of 1.9 was really needed for the case $\lambda = \mu(T) = \aleph_0$).

A substitute for 6.1 can be proven, namely: if $I(v_0)$ is minimal, $|I(v_0)| < \lambda$, $\lambda \geqq \mu'(T)$ and C is an algebraically closed set such that $I(C)$ contains a set of indiscernibles of power $\geqq \lambda$ then C is λ-saturated with respect to $I(v_0)$. Using this one can prove the conclusion of 6.5 for any $\lambda_1 \geqq \mu'(T)$.

Remark. As we learned after completing work, S. Shelah has proven 6.5 earlier than us and for $\lambda_1 \geqslant \kappa'(T)$, where $\kappa'(T) = \kappa(T)$ if $\mathrm{cf}\kappa(T) > \omega$ and $\kappa'(T) = \kappa(T)^+$ otherwise. His proof will appear in his book which is in preparation.

References

1. J. T. Baldwin, *Countable theories categorical in incountable powers*, Ph. D. Thesis, Simon Fraser Univ., 1971.

2. J. T. Baldwin, *On a theorem of Lachlan*, to appear.

3. J. T. Baldwin and A. H. Lachlan, *On strongly minimal sets*, J. Symbolic Logic **36** (1971), 79–96.

4. C. C. Chang, *A note on the two-cardinal problem*, Proc. Amer. Math. Soc. **16** (1965), 1148–1155.

5. V. Harnik, *Stable theories and related concepts*, Ph. D. Thesis, The Hebrew University of Jerusalem, 1971.

6. V. Harnik, *On the existence of saturated models of stable theories*, to appear in Proc. Amer. Math. Soc.

7. V. Harnik and J. P. Ressayre, *Prime extensions and categoricity in power*, Israel J. Math. **10** (1971), 172–185.

8. A. H. Lachlan, *A property of stable theories*, Fund. Math. **57** (1972), 9–20.

9. W. E. Marsh, *On ω_1 but not ω-categorical theories*, Ph.Thesis, Dartmouth College, 1966.

10. M. D. Morley, *Categoricity in power*, Trans. Amer. Math. Soc. **114** (1965), 514–538.

11. M. D. Morley and R. L. Vaught, *Homogeneous universal models*, Math. Scand. **11** (1962), 37–57.

12. J. P. Ressayre, *Sur les théories du premier ordre categoriques en un cardinal*, Trans. Amer. Math. Soc. **142** (1969), 481–505.

13. S. Shelah, *Categoricity of classes of models*, Ph. D. Thesis, The Hebrew University of Jerusalem, 1969.

14. S. Shelah, *Stable theories*, Israel J. Math. **7** (1969), 187–202.

15. S. Shelah, *Finite diagrams stable in power*, Ann. Math. Logic **2** (1970), 69–118.

16. S. Shelah, *Stability, the f.c.p. and superstability; model theoretic properties of formulas in first order theory*, Ann. Math. Logic **3** (1971), 271–362.

17. R. L. Vaught, *A Löwenheim-Skolem theorem for cardinals far apart*, The Theory of Models, North Holland, Amsterdam 1965, pp. 390–401.

DEPARTMENT OF MATHEMATICS
DARTMOUTH COLLEGE
HANOVER, NH. 03755 U.S.A.

Current address
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HAIFA
HAIFA, ISRAEL